Granični zadatak za linearne dif. j-ne

$x'' + x = 0$

a) $x'(0) = 0$, $x'(1) = 1$

b) $x'(0) = 0$, $x'(\pi) = 1$

c) $x'(0) = 0$, $x'(\pi) = 0$

$P(\lambda) = \lambda^2 + 1 = 0$

$\lambda_{1,2} = \pm i$

$x = c_1 \cos t + c_2 \sin t$

$x' = -c_1 \sin t + c_2 \cos t$

$$x'(0) = 0 \Rightarrow \quad \begin{cases} c_2 = 0 \\ -c_1 \sin 1 + \overset{0}{c_2} \cos 1 = 1 \end{cases} \quad \Leftarrow x'(1) = 1$$

$$c_1 = -\frac{1}{\sin 1}$$

$\Rightarrow x = -\frac{1}{\sin 1} \cos t$ - jedinstveno rješenje

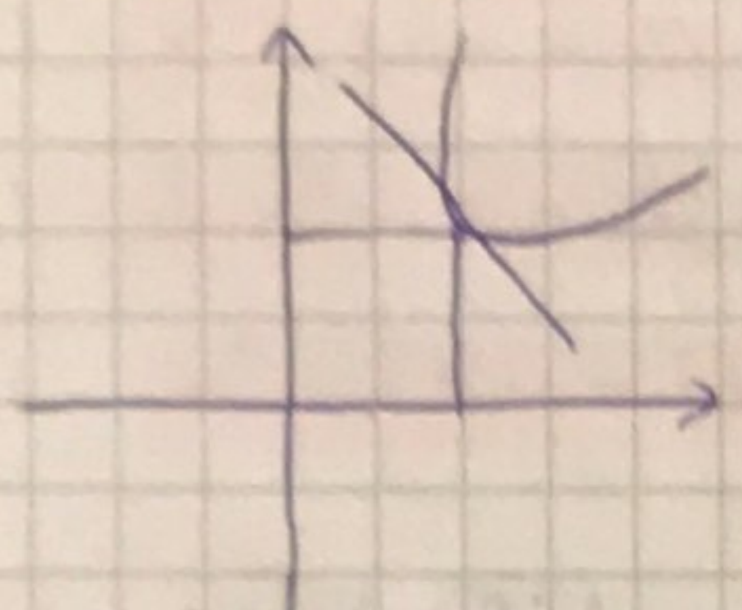
b) $x'(0)=0 \Rightarrow c_1=0$

$x'(\pi)=1 \Rightarrow c_2=1$ $\Rightarrow$ nema rješenje

c) $x'(0)=0 \Rightarrow c_2=0$

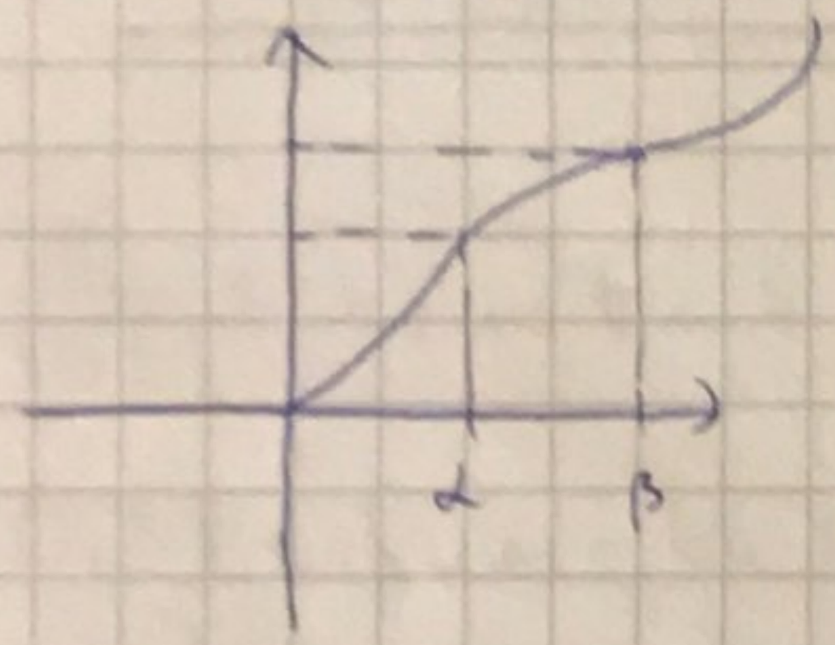
$x'(\pi)=0 \Rightarrow c_1=0$ $\Rightarrow$ $x=c_1\cos t, \quad c_1\in\mathbb{R}$

$$L(x)=x^{(n)}+a_1(t)x^{(n-1)}+\dots+a_n(t)x=b(t), \quad a_i, b\in C(I)$$



$x(t_0)=x_0$

$x'(t_0)=x_0'$



$x(\alpha)=x_1$

$x(\beta)=x_2$

Pretpostavimo da je $L(x)=b(t)$

$U_1(x)=M_{11}x(\alpha)+\dots+M_{1n}x^{(n-1)}(\alpha)+N_{11}x(\beta)+\dots+N_{1n}x^{(n-1)}(\beta)=\gamma_1$

$\vdots$

$U_n(x)=M_{n1}x(\alpha)+\dots+M_{nn}x^{(n-1)}(\alpha)+N_{n1}x(\beta)+\dots+N_{nn}x^{(n-1)}(\beta)=\gamma_n$

$U=\begin{pmatrix}U_1\\ \vdots\\ U_n\end{pmatrix}, \quad \gamma=\begin{pmatrix}\gamma_1\\ \vdots\\ \gamma_n\end{pmatrix} \Rightarrow U(x)=\gamma$

$U: C^{n-1}(I)\to C(I)$ linearni operator

$$\begin{cases} L(x)=b(t)\\ U(x)=\gamma\end{cases}$$

$L(x)=0$

$U(x)=0$ $(*)$

$x\equiv 0$ jeste rješenje

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$x = c_1 \varphi_1(t) + \dots + c_n \varphi_n(t)$

$U(x) = 0$, sistem (n) j-na, nepoznate su $c_1, \dots, c_n$

$$U\left(\sum_{i=1}^{n} c_i \varphi_i(t)\right) = \sum_{i=1}^{n} c_i U(\varphi_i(t)) = 0$$

$$\Delta = \begin{vmatrix} U_1(\varphi_1) & \dots & U_n(\varphi_n) \\ \vdots & & \\ U_n(\varphi_1) & \dots & U_n(\varphi_n) \end{vmatrix} = 0$$

Lema: Granični zadatak $(*)$ ima netrivijalno rješenje akko je $\Delta = 0$

$L(x) = b(t)$

$U(x) = \gamma$

$x = y + v \quad , \quad U(v) = \gamma$

$$L(x) = L(y+v) = \underbrace{L(y)}_{=0} + \underbrace{L(v)}_{=b(t)} = b(t)$$

$$U(x) = U(y+v) = \underbrace{U(y)}_{=0} + \underbrace{U(v)}_{=\gamma} = \gamma$$

$\Rightarrow \quad L(y) = 0$

$\qquad U(y) = 0$

Teorema: Ako homogeni granični zadatak $(*)$ ima samo trivijalno rješenje, tada odgovarajući nehomogeni granični zadatak ima jedinstveno rješenje.

$$x = \int_{\alpha}^{\beta} G(t,s) b(s) \, ds \quad - \quad \text{Grinova funkcija}$$

n=2 $\quad x'' + a_1(t)x' + a_2(t)x = 0$

$$\begin{cases} \alpha_0 x'(\alpha) + \alpha_1 x(\alpha) = 0 \\ \beta_0 x'(\beta) + \beta_1 x(\beta) = 0 \end{cases}$$

$$G(t,s) = \begin{cases} \dfrac{x_1(s)x_2(t)}{W(s)}, & \alpha \le s \le t \\ \dfrac{x_1(t)x_2(s)}{W(s)}, & t \le s \le \beta \end{cases} = \begin{cases} \dfrac{x_1(t)x_2(s)}{W(s)}, & \alpha \le t \le s \\ \dfrac{x_1(s)x_2(t)}{W(s)}, & s \le t \le \beta \end{cases}$$

1. Koristeći Grinovu funkciju naći granični zadatak:

$x'' + x' = f(t)$

$x'(0) = 0, \quad x(+\infty) = 0$

$x(t) = 0, \quad t \to +\infty$

$x'' + x' = 0$

$P(\lambda) = \lambda^2 + \lambda = 0$

$\lambda(\lambda+1) = 0$

$\lambda = 0 \lor \lambda = -1$

$x = c_1 + c_2 e^{-t}, \quad x' = -c_2 e^{-t}$

$x'(0) = 0 \Rightarrow c_2 = 0 \Rightarrow x_1(t) = 1$

$x(+\infty) = 0 \Rightarrow c_1 + c_2 e^{-t} \xrightarrow[t\to\infty]{} 0$

$c_1 = 0$

$x \equiv 0 \Rightarrow \exists!\, G(t,s)$

$x_2(t) = e^{-t}$

$x_1(t) = 1$

$\Rightarrow W(t) = \begin{vmatrix} 1 & e^{-t} \\ 0 & -e^{-t} \end{vmatrix} = -e^{-t} \neq 0, \ \forall t$

$$G(t,s)=\begin{cases}\frac{1\cdot e^{-t}}{-e^{-s}}, & 0\le s\le t\\ \frac{1\cdot e^{-s}}{-e^{-s}}, & t\le s<+\infty\end{cases}=\begin{cases}-e^{s-t}, & 0\le s\le t\\ -1, & t\le s<+\infty\end{cases}$$

$$x=\int_0^{+\infty}G(t,s)f(s)ds=\int_0^t -e^{s-t}f(s)ds-\int_t^{+\infty}f(s)ds$$

2. $t^2x''+tx'-x=f(t)$

$x(1)=0$, $x(t)$ - ograničeno za $t\to\infty$

3. $x''-x=f(t)$

$x(t)$ - ograničeno za $t\to\pm\infty$ (može biti ograničeno kada je konstanta 0)

4. $t^2x''-2x=f(t)$

$x(1)=0$, $x(2)+2x'(2)=0$

5. Koristeći Grinovu funkciju naći rješenje zadatka

$x''+x=t$, $x(0)=x(\frac{\pi}{2})=0$.

R: 2. $t^2x''+tx'-x=f(t)$

$x=t^\alpha$

$x'=\alpha t^{\alpha-1}$

$x''=\alpha(\alpha-1)x^{\alpha-2}$

$t^2\alpha(\alpha-1)x^{\alpha-2}+t\alpha t^{\alpha-1}-t^\alpha=0$

$\alpha(\alpha-1)+\alpha-1=0$

$(\alpha-1)(\alpha+1)=0$

$\alpha=1,\ \alpha=-1$

$x_1=t$

$x_2=\frac{1}{t}$

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moramo dobiti jedinstveno rješenje kako bi Grinova f-ja imala Košijev zadatak

$$x = C_1 t + C_2 \frac{1}{t}$$

$$x(1) = C_1 + C_2 = 0 \Rightarrow C_1 = -C_2$$

$$x(t) = C_1 t + C_2 \frac{1}{t} \xrightarrow{t \to \infty} C_1 = 0$$ - uzimamo da je nula tako ne bi imali $C_1 \cdot \infty$.

$$C_2 = 0$$

$$x_1: \left. \begin{matrix} C_1 t + C_2 \frac{1}{t} \\ C_1 = -C_2 \end{matrix} \right\} \begin{matrix} C_1 = 1 \\ C_2 = -1 \end{matrix} \qquad x_1 = t - \frac{1}{t}$$

$$x_2: \left. \begin{matrix} C_1 t + C_2 \frac{1}{t} \\ C_1 = 0 \end{matrix} \right\} \begin{matrix} C_2 = 1 \\ x_2 = \frac{1}{t} \end{matrix}$$

$$G(s,t) = \begin{cases} \dfrac{x_1(s)x_2(t)}{W(s)}, & 1 \le s \le t \\ \dfrac{x_1(t)x_2(s)}{W(s)}, & t \le s < \infty \end{cases}$$

$$W(s) = \begin{vmatrix} s - \frac{1}{s} & \frac{1}{s} \\ 1 + \frac{1}{s^2} & -\frac{1}{s^2} \end{vmatrix} = -\frac{1}{s} + \cancel{\frac{1}{s^3}} - \frac{1}{s} - \cancel{\frac{1}{s^3}} = -\frac{2}{s}$$

$$G(s,t) = \begin{cases} \dfrac{(s - \frac{1}{s})\frac{1}{t}}{-\frac{2}{s}}, & 1 \le s \le t \\ \dfrac{(t - \frac{1}{t})\frac{1}{s}}{-\frac{2}{s}}, & t \le s < \infty \end{cases}$$

$$G(s,t) = \begin{cases} \dfrac{s^2 - 1}{-2t}, & 1 \le s \le t \\ -\frac{1}{2}(t - \frac{1}{t}), & t \le s < \infty \end{cases}$$

$$x = \int_1^{\infty} G(s,t) f(s)\, ds = \int_1^t \frac{s^2-1}{-2t} f(s)\, ds - \int_t^{+\infty} \frac{1}{2}(t - \frac{1}{t}) f(s)\, ds$$

za $f(s) = s$: $$x = -\frac{1}{2t} \int_1^t (s^3 - s)\, ds - \frac{1}{2}(t - \frac{1}{t}) \int_t^{+\infty} s\, ds =$$

$$= -\frac{1}{2t} \left(\frac{s^4}{4} - \frac{s^2}{2} \right) \Big|_1^t - \frac{1}{2}(t - \frac{1}{t}) \cdot \frac{s^2}{2} \Big|_t^{+\infty} = \ldots$$

5. $x'' + x = t$

$x(0) = x(\frac{\pi}{2}) = 0$

$x = c_1 \cos t + c_2 \sin t$

$x(0) = 0 \Rightarrow c_1 = 0 \Rightarrow x_1(t) = \sin t$

$x(\frac{\pi}{2}) = 0 \Rightarrow c_2 = 0 \Rightarrow x_2(t) = \cos t$

$$W(t) = \begin{vmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{vmatrix} = -1$$

$$G(t,s) = \begin{cases} -\sin s \cos t, & 0 \le s \le t \\ -\sin t \cos s, & t \le s \le \frac{\pi}{2} \end{cases}$$

$$x = \int_0^{\pi/2} G(t,s)\, s\, ds = \int_0^t -\sin s \cos t\, s\, ds + \int_t^{\pi/2} -\sin t \cos s \cdot s\, ds =$$

$$= -\cos t \int_0^t s \cos s\, ds - \sin t \int_t^{\pi/2} s \cos s\, ds = \dots =$$

$$= -\cos t\,(-s\cos s + \sin s)\Big|_0^t - \sin t\,(s \sin s + \cos s)\Big|_t^{\frac{\pi}{2}} = t - \frac{\pi}{2}\sin t$$

3. $x'' - x = f(t)$

$P(\lambda) = \lambda^2 - 1 = 0$

$\lambda = \pm 1$

$x = c_1 e^{-t} + c_2 e^{t}$

$c_1 e^{-t} + c_2 e^{t} \xrightarrow{t \to +\infty} \quad c_2 = 0$

$c_1 e^{-t} + c_2 e^{t} \xrightarrow{t \to -\infty} \quad c_1 = 0$